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MINIMAL DESIGN OF SANDWICH AXISYMMETRIC PLATES OBEYING MISES CRITERION

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Abstract—A method for solving the non-linear problem of minimum volume design of sandwich axisymmetric plates obeying the Mises criterion is presented; it is a version of the statical method in which use is made of techniques of the calculus of variations. These techniques are extended for application to the case of unbounded minimal solutions. Several specific examples are worked out for loadings in which the applied forces point to the same direction and support is provided at the inner and outer edges only. It is shown that the characteristic features of the minimal designs are independent of other peculiarities of such loadings, when the plate is supported at one edge only. The results of the paper are compared to those of similar cases of plates obeying the Tresca criterion.

1. INTRODUCTION

THE problem of minimal design of structures is defined as the determination of a design, specified to within a variable parameter t, which is safe under a given loading and minimizes some structural property. For minimal plastic design, with which this paper is exclusively concerned, a design is said to be safe when the given loading or a multiple of it by a factor greater than one is a yield point loading.

With t the thickness of the face sheets of a sandwich plate and the volume of the face sheets the property to be minimized, Drucker and Shield [1] established that the minimal design is the one admitting a mode of collapse for which the rate of energy dissipation in the face sheets is constant everywhere. Mróz [2] showed further that all designs admitting modes of collapse for which the rate of energy dissipation is constant on the surface of solid plates correspond to extrema of the volume.

On the basis of the above theorems a number of minimal designs were obtained for full axisymmetric [1, 3, 4], circular asymmetric [5] and elliptical [6] sandwich plates. For solid plates, full [3] and annular [7] designs which are locally minimal were also determined.

In all these applications it was tacitly assumed that a design does exist which actually attains the minimum volume. It is known however that in many problems this is not the case because no stress field§ can be found that satisfies everywhere both the statical

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§ Here and in the following by "stress" is meant "generalized stress".

requirements and the Drucker–Shield theorem. The difficulty was artificially circumvented with the formation of flanges or ribs of infinite thickness and vanishing width. Such a situation was encountered from the very inception of the method [1, 8] and in later applications [9]. But in plates it was met only recently, because of the small number of specific problems worked out by means of the Drucker–Shield theorem. In fact the problem arose only with the introduction of the statical method of minimum volume design [10–12].

The statical method ignores completely kinematical requirements. By taking into consideration all statically admissible stress fields and by demanding the design to be such that the stress point lie always on the yield curve it expresses the volume as a functional of the stress over the domain of the plate. The minimum volume design then reduces to a problem of the calculus of variations.

With safety defined by the Tresca yield criterion the volume functional for sandwich plates is linear. Megarefs [10–12] exploited this linearity by the technique of stress variation. This approach proved powerful enough to determine the minimal designs in all cases of full or annular axisymmetric sandwich plates for one directional loading and any conditions of edge support. Moreover it showed that the need for the formation of flanges or ribs is a natural consequence of the minimal requirements; for this reason these devices were called design constraints. What is more important still, the technique of stress variation revealed that although minimal designs requiring design constraints cannot be constructed, they can be approximated (in principle, to any desired degree) by what will be termed optimal designs defined by a sequence of statically admissible stresses one component of which is allowed to exceed in absolute value all bounds over an infinitesimal sub-domain of the plate. Thus the design constraints can be interpreted as relaxation of the continuity and boundedness condition imposed on the statically admissible stresses.

When the volume functional is not linear, it seems that the statical method has to resort to the techniques of the calculus of variation. This paper represents such an extension of the statical method to full and annular axisymmetric plates obeying the Mises criterion. The problem is solved in its complete generality. Minimal and optimal designs are determined by means of variational techniques and the results are readily applicable, without any further theoretical consideration, in the case of simple loadings, i.e. when the applied transverse forces are directed in the same sense and support is provided only at the edges. This is illustrated on a number of specific examples. Comparisons with the corresponding plates obeying Tresca's criterion are made from which it is seen that design constraints, which here too emerge as a natural consequence of the minimal requirements, appear in the same cases for both classes of plates.

The mathematical formulation in this paper is similar to that of [4]. This results from the fact that both papers treat fundamentally the same problem and use the same well-known parametrization of the Mises yield criterion. They differ however completely in their approach and scope, as the present paper is based on the statical method and not the Drucker–Shield theorem, encompasses annular plates as well with various support conditions and a wider range of loadings, and defines not only minimal but also optimal designs.

2. FORMULATION OF THE PROBLEM

Consider an annular plate of sandwich construction under the action of axisymmetric loads normal to its middle plane. Let a and b denote the radii of the inner and outer edges respectively,† A the area of the middle plane enclosed by the two edges and M_r, M_θ the radial and circumferential principal bending moments. The volume to be minimized is that of the face sheets

$$\int_A t(r)\,dA$$

where $t(r)$ is the unknown variable thickness of the face sheets, kept at a constant distance h by the inert core. If $f(M_r, M_\theta) = M_y$ represents the adopted yield criterion, where $M_y = \frac{1}{2}ht\sigma_y$ is the yield moment of the sandwich section of the plate, then $t(r)$ must be everywhere proportional to $f(M_r, M_\theta)$. The problem is then equivalent to the minimization of the functional

$$J = \int_a^b f(M_r, M_\theta) r\,dr. \tag{1}$$

In accordance with the statical approach to minimal design, only those stresses $(\mathbf{Q}) = (M_r, M_\theta)$ satisfying the statical requirements will be admitted. Thus if $T(r)$ denotes the shear force, then the admissible stresses are such that:

$$\frac{d}{dr}(rM_r) - M_\theta = rT \quad \text{in } [a, b] \tag{2a}$$

M_r is continuous in A and meets the statical boundary conditions at $r = a$ and $r = b$; (2b)

M_θ is bounded in A. (2c)

If it is assumed that the plate is supported over precisely one circle, $T(r)$ is completely specified by the loading.‡ Moreover, the loading will be restricted so that the shear T determined by it does not vanish except over discrete circles.§

The extremal condition for (1) as well as the natural boundary conditions are easily obtained by submitting the volume functional to variation. If δM_r and δM_θ represent admissible infinitesimal stress variations, the volume variation, to the lowest order, is given by

$$\delta J = \int_a^b \left(\frac{\partial f}{\partial M_r} \delta M_r + \frac{\partial f}{\partial M_\theta} \delta M_\theta \right) r\,dr. \tag{3}$$

Moreover, the stress variations δM_r and δM_θ must, by (2a) and the assumptions on T, satisfy the relation

$$\frac{d}{dr}(r\delta M_r) = \delta M_\theta. \tag{4}$$

† For a full plate, a is zero.

‡ In the terminology of the statical method such plates are referred to as definite. Indefinite plates, which are supported over more than one circle, are treated in Section 6.

§ If the shear vanishes over a finite subdomain of $[a, b]$ some singularities may arise which have been investigated in [13] and can be excluded here.

Substitution of (4) into (3) yields, after integrating by parts and setting $\delta J = 0$,

$$\int_a^b \left\{ \frac{\partial f}{\partial M_r} - \frac{\mathrm{d}}{\mathrm{d}r} \frac{(r\partial f)}{\partial M_\theta} \right\} \delta M_r r \,\mathrm{d}r + \left[r^2 \frac{f}{M} M_r \right]_a^b = 0 \tag{5}$$

for all admissible variations δM_r, δM_θ, which means for δM_r continuous in A and vanishing at the edges at which a statical condition is imposed. The latter condition, if applicable at both edges, entails the relation

$$\int_a^b \delta M_\theta \,\mathrm{d}r = 0 \tag{6}$$

obtained from (4) by integration between the limits a and b. Furthermore since the yield criterion $f(M_r, M_\theta)$ is bounded, symmetric and homogeneous of order one in the arguments M_r and M_θ it follows that

$$\frac{\partial f}{\partial M_r} = \frac{2M_r - M}{2f}; \qquad \frac{\partial f}{\partial M_\theta} = \frac{2M - M_r}{2f} \tag{7}$$

are bounded and continuous, except for $M_r = M_\theta = 0$.

Assuming that $M_r = 0$ at both edges, (5) yields the Euler necessary condition of extremality

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\partial f}{\partial M_\theta} \right) - \frac{\partial f}{\partial M_r} = 0 \tag{8}$$

from which, and the boundedness of $\partial f / \partial M_r$, follows that $\partial f / \partial M_\theta$ is a continuous function in all its arguments r, M_r, M_θ.

The natural boundary conditions associated with (8) are easily obtained from (5). They state that if M_r is not prescribed at the outer edge, then

$$r \frac{\partial f}{\partial M_\theta} = 0 \quad \text{at } r = b \tag{9}$$

and if it is not prescribed at the inner edge

$$r \frac{\partial f}{\partial M_\theta} = 0 \quad \text{at } r = a. \tag{10}$$

Compactly, the statical as well as the natural boundary conditions can be put in the form

$$r \delta M_r \frac{\partial f}{\partial M_\theta}\bigg|_{a,b} = 0. \tag{11}$$

So far, conditions (8) and (11) have been shown only to be necessary for a minimal design. They can be shown to be also sufficient in the case of a sandwich plate obeying the Mises criterion, which is considered here. This statement is expressed more precisely in the following theorem the proof of which is given in the Appendix.

Theorem 1

If the yield moment $f(M_r, M_\theta)$ is a homogeneous function of the order one in both its arguments M_r and M_θ, and if $f(M_r, M_\theta)$ is continuously differentiable for all M_r, M_θ

except possibly $M_r = M_\theta = 0$, and if a statically admissible stress (M_r^*, M_θ^*) exists that satisfies (8) and (11) then this stress is unique and minimal, i.e. for this and only this stress the volume integral actually attains its minimum value.

3. MINIMAL DESIGNS

In this section it will be assumed that a stress can be found which satisfies all the statical requirements (2a)–(2c) and the minimal conditions (8), (11) everywhere in $[a, b]$. This means that the minimal stress does exist and that the minimal design, which actually attains the minimum volume, can be constructed. On this assumption detailed criteria for the determination of the minimal stress will be established.

As stated in the introduction the material of the face sheets is assumed to fail according to the Mises criterion; the function $f(M_r, M_\theta)$ is thus specified as

$$f = (M_r^2 - M_r M_\theta + M_\theta^2)^{\frac{1}{2}} \tag{12}$$

and is shown in Fig. 1.

Alternatively, (12) admits the parametric representation

$$\left.\begin{aligned} M_r &= f\left(\cos\alpha - \frac{1}{\sqrt{3}}\sin\alpha\right), \\ M_\theta &= f\left(\cos\alpha + \frac{1}{\sqrt{3}}\sin\alpha\right), \end{aligned}\right\} \tag{13}$$

where α is restricted to lie on the interval $[0, 2\pi)$.

With the representation (13), the derivatives $\partial f/\partial M_r$ and $\partial f/\partial M_\theta$ become functions of α alone, namely

$$\left.\begin{aligned} \frac{\partial f}{\partial M_r} &= \frac{1}{2}(\cos\alpha - \sqrt{3}\sin\alpha) \\ \frac{\partial f}{\partial M_\theta} &= \frac{1}{2}(\cos\alpha + \sqrt{3}\sin\alpha). \end{aligned}\right\} \tag{14}$$

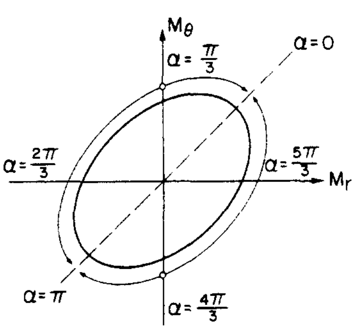

FIG. 1. The Mises yield curve. Directions of variation of α for increasing r.

Substitution of (14) into (8) yields

$$2\sqrt{3}\sin\alpha = r(\sin\alpha-\sqrt{3}\cos\alpha)\frac{d\alpha}{dr} \tag{15}$$

which admits the solutions:

$$\alpha \equiv 0(\pi) \tag{16}$$

and

$$r = k\exp[(\alpha\sqrt{3})/6]/|\sin\alpha|^{\frac{1}{2}} \tag{17}$$

where k is a positive constant.

M_r and $\partial f/\partial M_\theta$ must be continuous; however M_θ and $\partial f/\partial M_r$ are only required to be bounded. If $r = r_0$ is a circle of discontinuity in M_θ, then continuity of M_r and $\partial f/\partial M_\theta$ demand that $M_r(r_0) = 0$. But equation (13a) requires that either $f(r_0) = 0$ or $\alpha(r_0) = (\pi/3)(4\pi/3)$, and (14b) that $\cos(\alpha-\pi/3)$ be continuous everywhere. Therefore if $\alpha(r_0) = (\pi/3)(4\pi/3)$, then α is continuous at r_0. By (17) $\alpha = (\pi/3)(4\pi/3)$ is a local minimum for $r = r(\alpha)$; this condition can, therefore, be satisfied at only the edge $r = a$. Consequently f vanishes at $r = r_0$. The observations above may be summarized as follows: M_r, M_θ and f are continuous everywhere; discontinuities in α are permitted provided $f = 0$, and $\cos(\alpha-\pi/3)$ is continuous at the circle $r = r_0$ of discontinuity.

Circles over which M_r vanishes are termed inflexion circles.† Subdomains of the plate delimited by consecutive inflexion circles, by $r = a$ and the innermost inflexion circle, and by the outermost inflexion circle and the edge $r = b$ are called regions. The solutions (16) or (17) governing each region are called stress regimes or, simply, regimes. The regimes corresponding to $\alpha = 0, \pi$ are termed isotropic and those corresponding to $\alpha = 2\pi/3, 5\pi/3$ radial.

The regimes (16) and (17) determine a point on the Mises ellipse (12) corresponding to each circle with radius r of the plate. Examination of (17) shows that as α tends to either zero or π, r tends to infinity; consequently the isotropic regime cannot be approached in a continuous manner. The arrows in Fig. 1 indicate the direction of the variation of α for increasing radius.

Equations (2a), (13) and (15) may be combined to yield the linear equation

$$\frac{df}{d\alpha}-\frac{4\sqrt{3}}{3(\sqrt{3}\cot\alpha-1)}f+\frac{rT}{2\sin\alpha} = 0. \tag{18}$$

The solution of this equation, associated with each region, is given by

$$f(\alpha) = f_h(\alpha)\int_{\alpha_c}^{\alpha}\frac{H(\xi)}{f_h(\xi)}d\xi \tag{19}$$

where

$$f_h(\alpha) = \exp\left[\frac{-\alpha}{\sqrt{3}}\right]\Bigg/\left(\frac{1}{\sqrt{3}}\sin\alpha-\cos\alpha\right). \tag{20a}$$

$$H(\alpha) = -rT(r)/2\sin\alpha \tag{20b}$$

† For the terminology of the statical method, see [10–12].

and α_c is an inflexion circle.† Once the inflexion circles are located, the minimal solution is completely specified by (16), (17) and (19).

The number and location of the inflexion circles is not arbitrary. In particular, if the shear $T(r)$ has the same sign everywhere, then continuity of $\cos(\alpha-\pi/3)$ and equations (16), (17) require that neither $H(\alpha)$ nor $f_h(\alpha)$ change sign in $[a, b]$. Therefore, by (19), $f(\alpha)$ can vanish only once in $[a, b]$. The above observations are summarized in the following theorem.

Theorem 2

Let the loading be such that $T(r)$ is either non-negative or non-positive in $[a, b]$. Then there will be exactly one inflexion circle $r = c, a \leq c \leq b$.

4. OPTIMAL DESIGNS

This section deals with the case in which no stress can be found satisfying all statical and minimal conditions everywhere in $[a, b]$. The minimal design, therefore, cannot be constructed. At the best it can be approximated by a sequence of optimal designs defined by a sequence of statically admissible stresses $\{(\mathbf{Q}_n)\} = \{(M_{rn}, M_{\theta n})\}$ such that for each n the value J_n of the volume functional corresponding to the stress $(\mathbf{Q}_n)$ of the sequence is the absolute minimum among the values the volume functional assumes for all admissible stresses (M_r, M_θ) with $|M_\theta| \leq |M_{\theta n}|$. On this assumption detailed criteria will be established for the construction of an optimal stress sequence $\{(Q_n)\}$.

Denote by (M) the set of all statically admissible stresses and by (J) the set of the corresponding values of the volume functional. Since its elements are non-negative, (J) will have a greatest lower bound J_0. Consider next the subset (M^K) of (M) defined so that a stress (M_r, M_θ) is an element of (M^K) whenever (i) it is an element of (M), (ii) $|M_\theta| \leq K$ for all r in $[a, b]$ where K is an arbitrary positive number. Let J_0^K be the absolute minimum value of the volume functional over (M^K) and denote by (J^K) the set of these values for all values of K. It will be shown that if K tends to infinity

$$\lim_{K\to\infty} J_0^K = J_0. \tag{21}$$

Since J_0^K is non-increasing for increasing K and the set $\{J_0^K\}$ is bounded from below by J_0, it follows that as K tends to infinity the corresponding J_0^K converge to some limit J^*, where $J^* \geq J_0$. From the notion of greatest lower bound also follows that it is possible to construct a sequence $\{(\mathbf{Q}_s)\} = \{(M_{rs}, M_{\theta s})\}$ from elements of (M) such that the sequence of the corresponding values $\{J_s\}$ of the volume functional converges to J_0 as s tends to infinity. Therefore there exists a number S such that for all $s \geq S$, $|J_s - J_0| < e$. But, by the construction of the sequence $\{(\mathbf{Q}_s)\}$ there exists a number K such that $|M_{\theta S}| \leq K$ for all r in $[a, b]$. Then $(M_{rS}, M_{\theta S})$ is an element of (M^K). Since $J_0 \leq J_0^K \leq J_N$, it follows that $|J_0^K - J_0| < e$ and the relation (21) has therefore been established. Moreover if the absolute minimal stress in M^K is denoted by $(M_r^{*K}, M_\theta^{*K})$ and K is given the values of a monotonically increasing sequence $\{K_n\}$ with $K_n \to \infty$ as $n \to \infty$, then an optimal stress sequence is defined

† That every minimal design must exhibit at least one inflexion circle is easily established. The only case that need be considered is the one in which no statical boundary conditions are prescribed. It follows then from (11) and (14b) that at the two edges α takes the values $5\pi/6$ or $11\pi/6$. In view of Fig. 1 it is evident, that whatever the combination these boundary values cannot be attained without a discontinuity in α; but it was shown that such a discontinuity corresponds to an inflexion circle.

as

$$\{(M_{rn}, M_{\theta n})\}_* = \{(M_r^{*K_n}, M_\theta^{*K_n})\}.$$

After these preliminary results it is possible to establish necessary and sufficient conditions for the determination of the optimal stress sequence. They are formulated in the following two theorems the proofs of which are given in the Appendix.

Theorem 3

Suppose a minimal stress in (M) does not exist. If K is chosen sufficiently large, then necessary conditions that (M_r^K, M_θ^K) be minimal stress in (M^K) are:

(i) $|M_\theta^K| = K$ for $a \leq r < a+\varepsilon$;

(ii) equation (8) holds for $a+\varepsilon \leq r \leq b$;

(iii) $M_r^K \dfrac{\partial f}{\partial M_\theta}\Big|_{r=b} = 0$;

(iv) $M_r(a) = 0$;

(v) $\dfrac{\partial f}{\partial M_\theta}\Big|_{\substack{a+\varepsilon \\ \varepsilon \to 0}} \to \pm 1$ whenever $M_\theta \gtrless 0$ on $[a, a+\varepsilon)$

where $\varepsilon \to 0$ as $K \to \infty$.

Theorem 4

If in (M^K), there exist a (M_r^K, M_θ^K) satisfying conditions (i)–(v) of Theorem 3, then (M_r^K, M_θ^K) is the minimal stress in (M^K) for K sufficiently large.

As a consequence of condition (v) in the above theorems, and of (14b), α tends to $(\pi/3)(4\pi/3)$ as r tends to a. This result is also obtained from (13a) when $M_r(a) = 0$ and $f(a) \neq 0$. By (19), $f(\alpha)$ tends to infinity, but M_r remains bounded as α tends to $(\pi/3)(4\pi/3)$. However the choice $\alpha_a = \pi/3$ or $4\pi/3$ must be considered with both $f \geq 0$ and equilibrium in the vicinity of $r = a$. The latter condition is equivalent to $M_r(a^+) \gtrless 0$ when $M_\theta(a^+) \to \pm\infty$. It follows, therefore, that the optimal design satisfies (19); the boundary condition at the inner edge is the same as for the minimal design, i.e. $f = 0$ or $\alpha = (\pi/3)(4\pi/3)$, but the boundedness condition on f is relaxed to allow f to blow up in the vicinity of the inner edge.

5. EXAMPLES OF DEFINITE PLATES

1. Consider an annular plate clamped at its inner edge $r = a$ and free at its outer edge $r = b$. Let the loading be arbitrary, provided the shear stress $T(r)$ is everywhere of the same sign. In particular any uni-directional axisymmetric load will suffice. Without loss in generality, it will be assumed that $T(r) \geq 0$ for $a \leq r \leq b$. At $r = a$, $\partial f/\partial M_\theta$ must vanish. It follows from (14b) that $\alpha_a = 5\pi/6$; the choice $\alpha_a = 11\pi/6$ leads to a negative design f. Since $f(b) = 0$, the assumption of another inflexion circle violates Theorem 2. Consequently there can be but one region, namely, the entire plate. The minimal solution, from (19), is

$$f(\alpha) = \frac{e^{-\alpha\sqrt{3}}}{(1/\sqrt{3})\sin\alpha - \cos\alpha} \int_{\alpha_h}^{\alpha} e^{\xi\sqrt{3}} \left(\frac{1}{\sqrt{3}}\sin\xi - \cos\xi\right)\left(\frac{-rT(r)}{2\sin\xi}\right) d\xi \tag{22}$$

where $5\pi/6 \leq \alpha \leq \alpha_b < \pi$. The constant k in (17) is determined from the condition $\alpha_a = 5\pi/6$. The value α_b which can be interpreted as a parameter of the plate geometry, is also determined from (17):

$$\left(\frac{b}{a}\right)^2 = \exp\left[\frac{\sqrt{3}}{3}\left(\alpha_b - \frac{5\pi}{6}\right)\right]/(2 \sin \alpha_b). \tag{23}$$

A qualitative sketch of the minimal stress is given in Fig. 2.

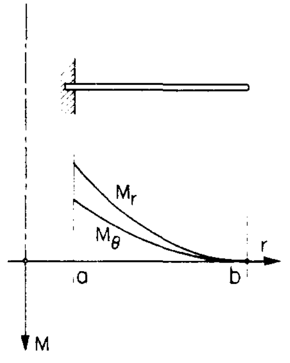

FIG. 2. Minimal stress for plate with clamped inner edge, free outer edge.

2. Consider an annular plate constrained against rotation at its inner edge $r = a$, and clamped at its outer edge $r = b$. If one directional downward loading is assumed, then $T(r) \leq 0$ for $a \leq r \leq b$. At each edge α must be either $5\pi/6$ or $11\pi/6$. By Theorem 2, there must be precisely one inflexion circle, say, at $r = c, a < c < b$. The condition $f \geq 0$ requires $\alpha_a = 11\pi/6, \alpha_b = 5\pi/6$. Denoting α_c^- and α_c^+ as the left and right hand limits of α at the discontinuity $r = c$, it follows from the continuity of $\cos(\alpha - \pi/3)$ that

$$\alpha_c^- = \frac{2\pi}{3} - \alpha_c^+ \tag{24}$$

where $11\pi/6 < \alpha_c^- < 2\pi$ and $\pi/3 < \alpha_c^+ + < 5\pi/6$. Also from (17), c satisfies

$$\left(\frac{c^2}{a}\right) = -e^{(\sqrt{3}/18)(6\alpha_c^- - 11\pi)}/(2 \sin \alpha_c^-) \tag{25a}$$

$$\left(\frac{b}{c}\right)^2 = 2e^{(\sqrt{3}/18)(5\pi - 6\alpha_c^+)} \sin \alpha_c^+. \tag{25b}$$

After eliminating α_c^- in (25) and using (24),

$$\left(\frac{b}{a}\right)^2 = -2e^{(\sqrt{3}/9)(4\pi - 6\alpha_c^+)}/(1 + \sqrt{3} \cot \alpha_c^+) \tag{26}$$

is obtained. The value α_c^+ determined from (26) may be taken as a given geometry parameter of the plate. The design f is now easily determined by (19) for each of the two regions, (a, c) and (c, b). The minimal stress is sketched in Fig. 3.

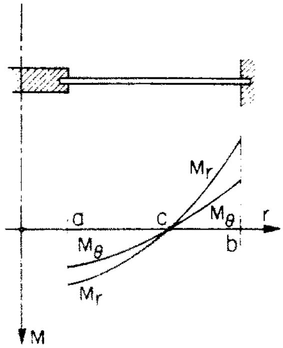

FIG. 3. Minimal stress for plate with rotationally constrained inner edge, clamped outer edge.

As a tends to zero, that is, as the annular plate tends to the full plate, the limiting ratio of c/b is obtained as

$$\lim_{\alpha \to 0} \left(\frac{c}{b}\right) = 0{\cdot}653 \tag{27}$$

The same critical value (27) was obtained by Eason [4] in the case of the full plate, clamped at $r = b$ and under uniform pressure. The analysis above, however, shows that the critical value is independent of the load. This type of invariance is not restricted to the Mises yield condition. Indeed Prager and Shield [5] and Megarefs [10] discovered this type of invariance for the Tresca yield criterion; the critical ratio c/b in this case is 2/3.

3. Consider the full plate clamped at its edge $r = b$ under the same type of loading as above. The boundary condition at $r = b$ is again $\alpha_b = 5\pi/6$. Isotropy at the center of the plate requires $\alpha_0 = 0$. Theorem 2 again requires an inflexion circle at, say, $r = c$. The condition $\alpha_0 = 0$ requires, by (16), that $\alpha \equiv 0$ in the region $0 \leq r < c$. From the jump condition, it follows that $\alpha_c^+ = 2\pi/3$. Upon substitution of $\alpha_b = 5\pi/6$ and $2\pi/3$ into (17) it is determined that

$$\frac{c}{b} = 0{\cdot}653 \tag{27'}$$

which is precisely (27). Consequently it may be concluded that the solution for the full plate is the limiting solution of the annular plate constrained at its inner edge.† The minimal stress is sketched in Fig. 4.

4. Consider an annular plate, free at its inner edge $r = a$, and simply supported at its outer edge $r = b$. If the loading is such that $T(r) \leq 0$ for $a \leq r \leq b$, it will follow from Theorem 2 that no interior inflexion circle may develop. The boundary conditions are then $f(\alpha_b) = 0$ and $\alpha_a = \pi/3$; the choice $\alpha_a = 4\pi/3$ leads to a solution f which either is negative or violates equilibrium near $r = a$. The value α_b, $0 < \alpha_b < \pi/3$, may be taken as a parameter defining the plate geometry, since by (17)

$$\left(\frac{b}{a}\right)^2 = \sqrt{3}\exp\left[\frac{\sqrt{3}}{3}\left(\alpha_b - \frac{\pi}{3}\right)\right] / (2\sin\alpha_b). \tag{28}$$

† Megarefs [10] showed that this is also true for the Tresca yield condition.

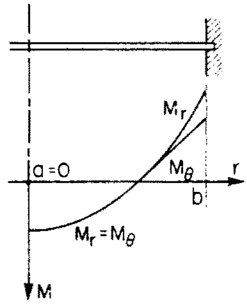

FIG. 4. Minimal stress for full plate with clamped periphery.

From (19), the design f satisfies

$$f = \frac{e^{-\alpha\sqrt{3}}}{1/\sqrt{3}\sin\alpha - \cos\alpha}\int_{\alpha_b}^{\alpha}\frac{1}{2\sqrt{3}}(-rT)(1-\cot\xi)e^{\xi\sqrt{3}}\,d\xi. \tag{29}$$

By (13) and (29) both f and M_θ tend to infinity as r tends to a, that is, as α tends to $\pi/3$; however M_r remains bounded at $r = a$.† Consequently a minimal design does not exist, but an optimal design does and is given by (29). The optimal stress is sketched in Fig. 5.

5. Consider a plate clamped at $r = b$, but otherwise the same as that of example (4). It is first assumed that α is continuous on the whole plate. At $r = b, \alpha = (5\pi/6)(11\pi/6)$. The boundary condition at the edge $r = a$ is

$$f(\alpha_a) = 0, \tag{30}$$

since from (17) the choice $\alpha_a = (\pi/3)(4\pi/3)$ is inconsistent with the continuity assumption on α and the boundary condition at $r = b$. From (19) and (30), the solution is given by

$$f = f_h(\alpha)\int_{\alpha_a}^{\alpha}\frac{H(\xi)}{f_h(\xi)}\,d\xi. \tag{31}$$

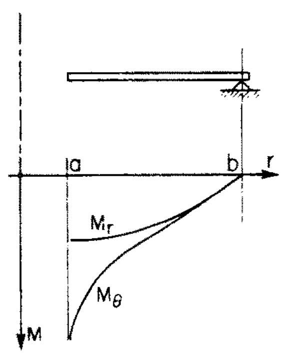

FIG. 5. Optimal stress for plate with free inner edge, supported outer edge.

† Megarefs [11], in his examination of this plate for the Tresca yield condition, found that the optimal design possessed a similar design constraint at the inner edge. The case treated here corresponds to the two cases (i) and (ii) of the five exhaustively investigated there.

The choice $\alpha_b = 11\pi/6$ leads to a negative f, therefore the proper condition is

$$\alpha_b = 5\pi/6. \tag{32}$$

The value α_a, determined by (17) is again a geometry parameter; it is calculated from the expression

$$\left(\frac{b}{a}\right)^2 = 2\exp\left[\frac{\sqrt{3}}{18}(5\pi-6\alpha_a)\right]\sin\alpha_a \tag{33}$$

where $\pi/3 < \alpha_a < 5\pi/6$. From (33), $(b/a)^2$ is a maximum when $\alpha_a = \pi/3$. Therefore α_a can satisfy (33) only if $(b/a)^2 \leq \sqrt{3}e^{(\sqrt{3})\pi/6} = 4{\cdot}263$, that is

$$\left(\frac{a}{b}\right) \geq 0{\cdot}485. \tag{34}$$

Therefore a minimal design exists and is given by (31) whenever the plate geometry satisfies (34).

Now let it be assumed that an inflexion circle occurs at $r = c, a < c < b$. Consider first the region delimited by $r = c$ and $r = b$. As before $\pi/3 < \alpha_c^+ < \alpha_b = 5\pi/6$ and $f(\alpha_c^+) = 0$. By the same reasoning as above, (31) and (33) will hold with a replaced by c, namely,

$$f(\alpha) = f_h(\alpha)\int_{\alpha_c^+}^{\alpha}\frac{H(\xi)}{f_h(\xi)}\,d\xi \qquad \text{for } \alpha_c^+ \leq \alpha \leq \frac{5\pi}{6} \tag{35}$$

$$\left(\frac{b}{c}\right)^2 = 2\exp\left[\frac{\sqrt{3}}{18}(5\pi-6\alpha_c^+)\right]\sin\alpha_c^+. \tag{36}$$

If $\alpha_c^+ \geq 2\pi/3$, the jump condition implies that $11\pi/6 < \alpha_c^- < 2\pi$. Since no other discontinuity in α is permissible, it follows from (17) that $\alpha_a = 4\pi/3$. But this choice requires $M_r(a^+) > 0$ while $M_\theta(a^+) \to -\infty$ which violates equilibrium. Therefore $\pi/3 < \alpha_c^+ < 2\pi/3$, and $\alpha_a = \pi/3$. From (17),

$$\left(\frac{c}{a}\right)^2 = \frac{\sqrt{3}}{2}\,\frac{\exp[(\sqrt{3}/9)(3\alpha_c^- - \pi)]}{\sin\alpha_c^-}, \qquad 0 < \alpha_c^- < \frac{\pi}{3}. \tag{37}$$

With (36), (37) and the jump condition

$$\alpha_c^+ = \frac{2\pi}{3} - \alpha_c^-, \tag{38}$$

it follows that

$$\left(\frac{b}{a}\right)^2 = \frac{3}{2}\left(\cot\alpha_c^- + \frac{1}{\sqrt{3}}\right)\exp\left[\frac{\sqrt{3}}{18}(12\alpha_c^- - \pi)\right]. \tag{39}$$

It can be shown that (b/a) is a minimum when $\alpha_c^- = \alpha_a = \pi/3$. Therefore α_c^- can be determined from (39) only when

$$\frac{a}{b} \leq 0{\cdot}485 \tag{40}$$

whence the optimal f is

$$f = f_h(\alpha)\int_{\alpha_c^-}^{\alpha}\frac{H(\xi)}{f_h(\xi)}\,d\xi \qquad \text{for } \alpha_c^- \leq \alpha \leq \frac{\pi}{3}. \tag{41}$$

The critical value $a/b = 0{\cdot}485$ is once again independent of the load. For $a/b \geq 0{\cdot}485$, there is a single region minimal solution. But as a/b becomes less than 0·485, the design changes abruptly into an optimal design with two regions and a singularity in f at the inner edge. As a/b approaches 0·485 from below, the two-regime solution approaches the single regime solution, however, non-uniformly. Using the Tresca yield criterion, Megarefs [10] observed the same type of phenomenon; the critical ratio of a to b was determined to be $\frac{1}{2}$. The minimal and the optimal stresses for the Mises criterion are sketched in Fig. 6.

6. INDEFINITE PLATES

In the preceding sections the plate was assumed to be definite; thus the shear force T was uniquely determined by the applied loads. In this section the design theory developed previously will be extended to indefinite plates, i.e. plates supported over more than one circle.

Let r_i ($i = 1, 2, \ldots, n$) denote the radii of the circles of support. Then the shear force is given by

$$T(r) = -\frac{1}{r}\left[\int_a^r p(\xi)\xi \, d\xi + \sum_{j=1}^{m} P_j r_j H(r-r_j) + \sum_{i=1}^{n} R_i r_i H(r-r_i)\right] \tag{42}$$

where $p(r)$ is the applied distributed load, P_j an applied ring load at $r = r_j$, R_i the reaction force at the circle of support $r = r_i$ and $H(r-r_i)$ the Heaviside operator with source at $r = r_i$. The n reactions R_i are related by only the force equation of the overall equilibrium; eliminating one of the reactions by this equation we can write the expression for the shear force in the form

$$rT(r) = rT^*(r) - \sum_{i=1}^{n-1} r_i R_i H(r-r_i). \tag{43}$$

The $n-1$ reactions R_i in (43) represent arbitrary independent parameters which must be determined from the minimal requirements.

If ρ_i are infinitesimal but arbitrary variations of the reactions R_i, then the infinitesimal variations δM_r and δM_θ must satisfy the equation

$$\frac{d}{dr}(r\delta M_r) = \delta M_\theta - \sum_{i=1}^{n-1} \rho_i r_i H(r-r_i). \tag{44}$$

Substitution of (43) into (3) yields

$$\delta J = \sum_{i=1}^{n-1} \rho_i r_i \int_a^b H(r-r_i)\frac{\partial f}{\partial M_\theta} r \, dr = 0 \tag{45}$$

where (8) and (11) were used. Since the $n-1$ ρ_i are independent, (45) can be satisfied if, and only if

$$\int_{r_i}^b \frac{\partial f}{\partial M_\theta} r \, dr = 0 \qquad \text{for } i = 1, \ldots, n-1. \tag{46}$$

Equations (45) supply the $n-1$ additional conditions required for the determination of the reactions R_i.

7. EXAMPLE OF INDEFINITE PLATE

Consider an annular plate, simply supported at both its inner edge $r = a$ and outer edge $r = b$, loaded with uniform pressure p_0. If R_1 and R_2 denote the reaction forces at $r = a$ and $r = b$ respectively, the shear may be expressed by

$$rT(r) = -\frac{p_0}{2}(r^2 - R) \tag{47}$$

where $R = a^2 - (2a/p_0)R_1$. If $a^2 - R \geq 0$, then by (47) $rT(r) \leq 0$ for $a \leq r \leq b$. By example (4), the solution must consist of a single regime with $0 < \alpha_b \leq \alpha \leq \pi/3$. For this regime, however, $\partial f/\partial M_\theta > 0$, and (46) cannot be satisfied. If $R \geq b^2$, then $T(r) \geq 0$. Again there may be but one regime with $\pi < \alpha_b < 4\pi/3$. For this regime $\partial f/\partial M_\theta < 0$ and again (46) cannot be satisfied. It may, therefore, be concluded that

$$a^2 < R < b^2. \tag{48}$$

From (47) and (48), it follows that T must vanish somewhere on (a, b). Let d be such that $T(d) = 0$.

There can be no inflexion circle at $r = c$ if $d \leq c < b$, for this would violate Theorem 2. On the other hand, the assumption of more than one inflexion circle, say c_1 and c_2 with $a < c_1 < c_2 < d$, also violates Theorem 2. Therefore there can be at most one interior inflexion circle; its radius c must satisfy $a < c < d$.

First assume that no interior inflexion circle develops. Now α_b lies in either $(5\pi/6, \pi)$ or $(11\pi/6, 2\pi)$; for any other value, (14b) and (17) would not allow $\partial f/\partial M_\theta$ to change sign in $[a, b]$. If α_b lies in $(5\pi/6, \pi)$, then (17) requires $\alpha_a < \alpha_b$. Then (19) yields $f \leq 0$ for $d \leq r \leq b$. Therefore

$$\frac{11\pi}{6} < \alpha_b < 2\pi. \tag{49a}$$

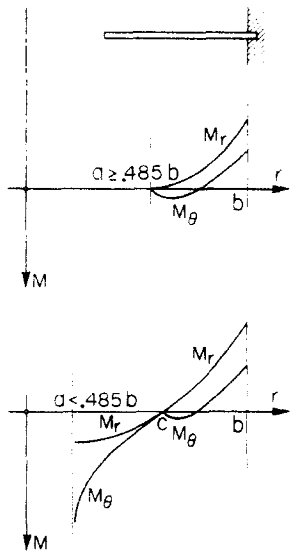

FIG. 6. Minimal and optimal stress for plate with free inner edge, clamped outer edge when $a/b \geq 0{\cdot}485$ and $a/b < 0{\cdot}485$ respectively.

By (49a) and (17)

$$\frac{4\pi}{3} \le \alpha_a < \frac{11\pi}{6} \tag{49b}$$

and

$$\left(\frac{b}{a}\right)^2 = \exp\left[\frac{1}{\sqrt{3}}(\alpha_b - \alpha_a)\right] \sin\alpha_a/\sin\alpha_b. \tag{50}$$

Moreover, substitution of (14b) and (15) into (46) yields

$$\int_{\alpha_a}^{\alpha_b} (\cot\alpha + \sqrt{3})(1 - \sqrt{3}\cot\alpha)\exp(\alpha/\sqrt{3})\,\mathrm{d}\alpha = 0. \tag{51}$$

It can be shown from (49a), (49b), (50) and (51) that

$$\frac{a}{b} \ge 0{\cdot}3095. \tag{52}$$

Therefore, for plate geometry satisfying (52) a minimal solution is found, viz.

$$f = f_h(\alpha)\int_{\alpha_b}^{\alpha} \frac{H(\xi)}{f_h(\xi)}\,\mathrm{d}\xi. \tag{53}$$

The values α_a, α_b are uniquely determined from (50) and (51) subject to (49a), (49b). The reaction parameter R appearing in the integrand of (53) is determined from the boundary condition $f(\alpha_a) = 0$.

For plate geometry in which $a/b < 0{\cdot}3095$, a single regime solution is not possible. If, however, an inflexion circle is assumed at $r = c$, arguments similar to those above will show that

$$\alpha_a = \frac{4\pi}{3};\qquad \frac{11\pi}{6} < \alpha_b < 2\pi;\qquad \alpha_c^+ < \frac{5\pi}{3};\qquad \alpha_c^+ = \frac{8\pi}{3} - \alpha_c^-. \tag{54}$$

Moreover, it follows from (17) that

$$\left(\frac{c}{a}\right)^2 = -\frac{\sqrt{3}}{2\sin\alpha_c^-}\exp\left[\frac{1}{\sqrt{3}}\left(\alpha_c^- - \frac{4\pi}{3}\right)\right], \tag{55a}$$

$$\left(\frac{b}{c}\right)^2 = \frac{\sin\alpha_c^+}{\sin\alpha_b}\exp\left[\frac{1}{\sqrt{3}}(\alpha_b - \alpha_c^+)\right], \tag{55b}$$

and

$$\left(\frac{b}{a}\right)^2 = -\frac{\sqrt{3}\sin\alpha_c^+}{\sin\alpha_b\sin\alpha_c^-}\exp\left[\frac{1}{\sqrt{3}}\left(\alpha_c^- + \alpha_b - \alpha_c^+ - \frac{4\pi}{3}\right)\right]. \tag{55c}$$

And from (46),

$$\int_{4\pi/3}^{\alpha_c^-} (\cot\alpha + \sqrt{3})(1 - \sqrt{3}\cot\alpha)e^{\alpha/\sqrt{3}}\,\mathrm{d}\alpha + \left\{\frac{\sin\alpha_c^+}{\sin\alpha_c^-}\exp\left[\frac{1}{\sqrt{3}}(\alpha_c^- - \alpha_c^+)\right]\right\} \times\left\{\int_{\alpha_c^+}^{\alpha_b} (\cot\alpha + \sqrt{3})(1 - \sqrt{3}\cot\alpha)e^{\alpha/\sqrt{3}}\,\mathrm{d}\alpha\right\} = 0 \tag{56}$$

must be satisfied. With (54) through (56) it can be shown that $(b/a)^2$ will be minimized when $\alpha_c^- = \alpha_c^+ = 4\pi/3$. Then

$$\left(\frac{b}{a}\right)^2 \geq -\frac{\sqrt{3}}{2\sin\alpha_b}\exp\left[\frac{1}{\sqrt{3}}\left(\alpha_b-\frac{4\pi}{3}\right)\right], \tag{57}$$

where

$$\int_{4\pi/3}^{\alpha_b}(\cot\alpha+\sqrt{3})(1-\sqrt{3}\cot\alpha)e^{\alpha/\sqrt{3}}\,d\alpha = 0. \tag{58}$$

The solution α_b of (58) is 6·039, whence (57) requires

$$\left(\frac{a}{b}\right) \leq 0{\cdot}3095. \tag{59}$$

For plate geometry satisfying (59), α_c^-, α_c^+, α_b and the inflexion circle $r = c$ are uniquely determined by (54), (55a), (55b) and (56). The optimal design determined by (19) is

$$f(\alpha) = \begin{cases} f_h(\alpha)\displaystyle\int_{\alpha_c^-}^{\alpha}\frac{H(\xi)}{f_h(\xi)}\,d\xi & \text{for } \dfrac{4\pi}{3} \geq \alpha \geq \alpha_c^- \\ f_h(\alpha)\displaystyle\int_{\alpha_b}^{\alpha}\frac{H(\xi)}{f_h(\xi)}\,d\xi & \text{for } \alpha_c^- \leq \alpha \leq \alpha_b. \end{cases} \tag{60}$$

The undetermined reaction R occurring in the integrand of (60) is determined by the hitherto unsatisfied condition, $f(\alpha_c^+) = 0$. The remarks made at the end of Example 5 of Section 4 apply here as well. Using the Tresca yield criterion, Megarefs [12] discovered the same kind of behavior: the critical ratio for that case was $a/b = 0{\cdot}326$. The minimal and optimal stresses for the Mises criterion are sketched in Fig. 7.

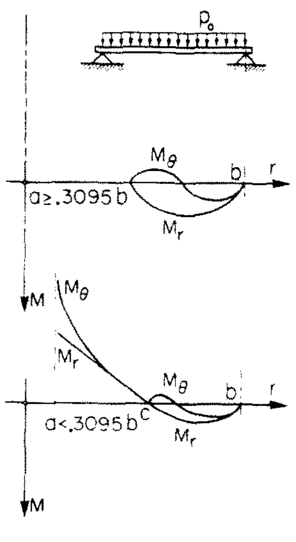

FIG. 7. Minimal and optimal stress for plate with both edges supported when $a/b \geq 0{\cdot}3095$ and $a/b < 0{\cdot}3095$ respectively.

8. CONCLUSION

It has been shown that the statical method, using techniques of the calculus of variations, can supply the necessary information for the complete determination of the minimum volume design in all cases of sandwich axisymmetric plates obeying the Mises criterion. Previously the method, using the technique of stress variation [10–12], proved equally successful for the Tresca criterion. It can therefore be said that the statical method has given the thorough and complete solution to the problem of minimum volume design of axisymmetric plates which had eluded investigation for more than a decade.

Most striking among the contributions of the statical method is the discovery that a minimal design, corresponding to a well defined stress in the closed interval $[a, b]$ does not always exist. When it does not, optimal solutions can be obtained which correspond to a sequence of admissible stresses and approximate the minimum volume, theoretically, to any desired degree. In the limit the absolute value of the circumferential component of the stress, and thus also the thickness of the face sheets, grow beyond all bounds over an infinitesimal subdomain adjoining the inner edge. The techniques of the calculus of variations were extended in this paper to deal with such a situation. As a result the design constraint emerged again as a natural and gradual consequence of the minimality requirements. Obviously, the assumptions of thin plate theory, on which the whole investigation is based, cease to be valid when the thickness of the face sheets exceeds a certain value and the problem must be then formulated anew. A great asset of the statical method is that both, it can determine the best possible design under the restriction of limited thickness and indicate the way for the proper reformulation of the problem.

Although the notion of existence has been used in connection with the above situation the paper was not intended to and does not contain any existence theorem proper. If the set of the admissible stresses was extended to comprise generalized elements the sequence of optimal stresses could be made to converge to a point of this space and thus to exist in the mathematical sense; the corresponding design however would continue to be unattainable and physically absurd. For this reason it was considered preferable to emphasize the physical side rather than the mathematical and exclude generalized elements.

All examples of the paper refer to simple loadings, i.e. loadings which consist of applied forces pointing in the same direction with reactions developing only at the edges. The solution is then substantially easier and, for definite plates, independent of any other peculiarities of the load. For more complicated loads further refinement of the theory is necessary.

There are several striking similarities between the minimal and optimal designs of sandwich axisymmetric plates obeying the Mises and the Tresca [10–12] criteria. For both criteria design constraints are required in the same cases and the critical value of the geometry parameter at which they appear differs little. For this critical value the minimal design turns into an optimal design and the pattern of the corresponding stress changes abruptly. The graph of the radial moment for the two criteria is of the same overall form; but the graph of the circumferential moment for the Mises criterion does not exhibit the discontinuities found under the Tresca criterion; it does never assume the constant value $M_\theta = 0$ and only exceptionally, in the central region, remains equal to M_r throughout a region. Furthermore in each region there exists only a single regime; regime circles, at which the radial regime turns into isotropic inside the same region were not discovered in the present investigation. In fact the isotropic regime under the Mises criterion either applies to all points of a region or it does not apply at all.

A further comparison between the minimal stresses under the two criteria could be made in terms of kinematical properties. Although kinematics do not appear explicitly or implicitly in this paper, or the statical method generally, it is not difficult to translate the statical results into kinematical language. It suffices to observe that $\partial f/\partial M_r$, $\partial f/\partial M_\theta$ represent to within a positive multiplicative constant the curvature rates $\dot{\kappa}_r$, $\dot{\kappa}_\theta$ of the minimal design at incipient plastic flow. The extremal equation (8a′) can then be interpreted as their compatibility conditions, the boundary conditions (11) are equivalent to the kinematical conditions at the edges for the various types of support, whereas (45) with $\partial f/\partial M_\theta = 1/r\, d\dot{w}/dr$ simply states that $\dot{w}(r_j) = \dot{w}(r_k)$ at all circles of support. Finally the Drucker–Shield theorem of constant energy dissipation is nothing else but Euler's theorem for homogeneous functions.

With the above correspondence it can be shown that even when a regime circle develops under the Tresca criterion, and the M_θ graph is thus discontinuous and quite different from its counterpart under the Mises criterion, as in the example of the indefinite plate, the strain rates are remarkably similar. For both criteria, $\dot{\kappa}_r$ and $\dot{\kappa}_\theta$ are continuous throughout each region; moreover $\dot{\kappa}_\theta$ is continuous over the entire plate whereas $\dot{\kappa}_r$ is discontinuous at the inflexion circle. The change in regime under the Tresca criterion is necessitated by the compatibility (or Euler) equation and the continuity requirement on the strains. Also when the isotropic regime prevails under the Tresca, but not under the Mises criterion the isotropic strain $\dot{\kappa}_r = \dot{\kappa}_\theta$ is never reached for either of them; for instance, in example 1 of definite plates the strain rate vector, for both criteria, starts with $\dot{\kappa}_\theta = 0$, $\dot{\kappa}_r < 0$ at $r = a$ and turns continuously until $\dot{\kappa}_r < \dot{\kappa}_\theta < 0$ and $r = b$. For the limiting case $a = 0$, i.e. the full plate, the prevailing regime is isotropic through the plate for both criteria, and so is the strain rate with $\dot{\kappa}_r = \dot{\kappa}_\theta$.

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APPENDIX

Proofs of theorems

Proof of Theorem 1. Define $\psi(\mathbf{Q}^0, \mathbf{Q}^1)$ by

$$\psi(\mathbf{Q}^0, \mathbf{Q}^1) = f(\mathbf{Q}^1) - f(\mathbf{Q}^0) - (Q_i^1 - Q_i^0)\frac{\partial f}{\partial Q_i}\bigg|_{\mathbf{Q}^0}, \tag{A1}$$

where $\mathbf{Q} = (M_r, M_\theta)$ is a stress, Q_i $(i = 1, 2)$ its components $Q_1 = M_r$, $Q_2 = M_\theta$, and the repeated index indicates summation. The function $f(Q)$ [defined by (12) and proportional to t since the plate is of sandwich construction] is convex and positive homogeneous of order one. In the coordinate system Q_1, Q_2, z the surface $z = f(\mathbf{Q})$ is conical having as cross sections $z =$ const. the convex Mises yield curves for varying t. The scalar ψ is then easily identified as the inner product of the directed chord from (Q_1^0, Q_2^0, z^0) to (Q_1^1, Q_2^1, z^1) and the gradient to the surface $z - f(\mathbf{Q}) = 0$ at (Q_1^0, Q_2^0, z^0). Since the surface is convex $\psi \geq 0$ everywhere.

Let now $\mathbf{Q}$ and $\mathbf{Q}^0$ be admissible stresses. Putting $\Delta\mathbf{Q} = \mathbf{Q} - \mathbf{Q}^0$, it follows from (A1) that

$$f(\mathbf{Q}^0 + \Delta\mathbf{Q}) - f(\mathbf{Q}^0) = \frac{\partial f}{\partial Q_i}\bigg|_{\mathbf{Q}^0} \Delta Q_i + \psi(\mathbf{Q}^0, \mathbf{Q}). \tag{A2}$$

The variation ΔJ of the volume functional corresponding to the admissible stress variation $\Delta\mathbf{Q}$ is obtained by integrating the first member of (A2) over the area A of the plate. Obviously $\Delta\mathbf{Q}$ satisfies the equilibrium condition (4). If $\mathbf{Q}^0$ satisfies the minimal conditions (8) and (11) it is easily shown that the integral of the first term in the second member of (A2) vanishes. Then, for all admissible $\mathbf{Q}$

$$\Delta J = \int_A \psi(\mathbf{Q}^0, \mathbf{Q})\, \mathrm{d}A \geq 0 \tag{A3}$$

and Q^0 is thus shown to be minimal.

It should be noted that the equality sign in (A3) holds only when $\psi = 0$ everywhere in $[a, b]$. From the conical form of the surface $z = f(\mathbf{Q})$ and the strict convexity of its cross section it is evident that ψ will vanish only if the two stress points $\mathbf{Q}^0$ and $\mathbf{Q}$ lie on the same generator, i.e. $\mathbf{Q} = vQ_0$ with $v > 0$. But $\mathbf{Q}$ is then in equilibrium not with the applied load but its v-tuple; it is, therefore, not statically admissible, and it can be concluded that the minimal stress is unique.

Proof of Theorem 3. Let (M_r^K, M_θ^K) be the absolute minimal stress in (M^K) where K is sufficiently large. On at least one interval, say $[a_1, b_1)$, $|M_\theta^K| = K$. For otherwise, (M_r^K, M_θ^K) would be required to satisfy the Euler equation (8); but this contradicts the assumption that an absolute minimal stress does not exist in (M). Without loss in generality, assume

$$\begin{aligned} M_\theta^K &= K \qquad \text{for } a < a_1 \leq r < b_1 \leq b \\ |M_\theta^K| &\leq K \qquad \text{for } a_0 \leq r < a_1. \end{aligned} \tag{A4a}$$

Moreover equilibrium (2a) requires

$$|M_r^K| \ll K \qquad \text{for } a_1 \le r < b_1 . \tag{A4b}$$

Consider now another stress $(\overline{M}_r^K, \overline{M}_\theta^K)$ and its difference from the previous stress $(\Delta M_r, \Delta M_\theta) = (\overline{M}_r^K - M_r^K, \overline{M}_\kappa^K - M_\theta^K)$ which, in the terminology of the statical method, is a stress variation. Let this stress variation, in particular, be a double pulse. Precise definition of this stress variation and its properties can be found in [10] and [11]. For the present purposes it will be described as follows:

$$\Delta M_r = \begin{cases} \dfrac{\xi_0}{r}(r-a_0) \\ \dfrac{\xi_0}{r}\varepsilon \\ \dfrac{\xi_0}{r}(b_1 - r) \end{cases} \qquad \Delta M_\theta = \begin{cases} \xi_0 < 0 & \text{in } [a_0, b_0) \\ 0 & \text{in } [b_0, a_1] \\ -\xi_0 & \text{in } [a_1, b_1] \end{cases} \tag{A5}$$

where $b_1 - a_1 = b_0 - a_0 = \varepsilon$.

Since volume must be finite it follows that ε is small for K sufficiently large. Using

$$\begin{aligned} \frac{\partial f}{\partial M_\theta} &= 1 + O(\varepsilon^2) \\ \frac{\partial f}{\partial M_r} &= -\frac{1}{2} + O(\varepsilon) \end{aligned} \qquad a_1 \le r < d_1 \tag{A6}$$

and the requirement that (M_r^K, M_θ^K) satisfy (8) for $a_0 \le r < a_1$, it follows that the cost variation is

$$\delta J = \xi_0 \left[a_1 \varepsilon \left(\frac{\partial f}{\partial M_\theta}\bigg|_{a_1^-} - 1 \right) - \frac{3}{4}\varepsilon^2 + O(\varepsilon^3) \right] < 0. \tag{A7}$$

The assumed design is not a minimum; consequently it may be concluded that $a_1 = a$, that is, M_θ may become large at the inner edge only. If no statical condition is imposed at $r = a$, a negative pulse applied at the inner edge will show that $M_\theta^K < K$ at the inner edge.

It remains to show that $\partial f/\partial M_\theta|_{r=a+\varepsilon}$ tends to one as ε tends to zero and K tends to infinity. Apply a double pulse variation of intensity $-\xi_0$ ($\xi_0 > 0$) with the first pulse on $[a, a+\varepsilon)$. By arguments similar to those above, it follows that

$$\delta J = \xi_0 a\varepsilon \left(\frac{\partial f}{\partial M_\theta}\bigg|_{a+\varepsilon} - 1 \right) + \xi_0 O(\varepsilon^2). \tag{A8}$$

Clearly $\delta J < 0$ unless $\partial f/\partial M_\theta|_{a+\varepsilon} \to 1$ as $\varepsilon \to 0$.

Proof of Theorem 4. From (A1) it follows that for any admissible variation ΔM_r, ΔM_θ, the variation of the volume integral is

$$\Delta J = \int_a^{a+\varepsilon} \left(\frac{\partial f}{\partial M_r}\Delta M_r + \frac{\partial f}{\partial M_\theta}\Delta M_\theta \right) r\,\mathrm{d}r + \int_{a+\varepsilon}^{b} \left(\frac{\partial f}{\partial M_r}\Delta M_r + \frac{\partial f}{\partial M_\theta}\Delta M_\theta \right) r\,\mathrm{d}r + \int_a^b \psi r\,\mathrm{d}r. \tag{A9}$$

Since the third integral in (A9) is non-negative, it suffices to show that

$$I_1 + I_2 > 0 \tag{A10}$$

where I_1 and I_2 are the first and second integrals respectively in (A9). Without loss in generality, assume $M_\theta^* = K \gg 0$ for $a \le r < a+\varepsilon$.

Using conditions (ii) and (iii), of the hypothesis, I_2 becomes

$$I_2 = -(a+\varepsilon)^2 \,\Delta M_r(a+\varepsilon) \frac{\partial f}{\partial M_\theta}\bigg|_{a+\varepsilon}. \tag{A11}$$

Since

$$\begin{aligned} \frac{\partial f}{\partial M_\theta} &= 1 + O(\varepsilon^2) \qquad a \le r < a+\varepsilon \\ \frac{\partial f}{\partial M_\theta} &= -\frac{1}{2} + O(\varepsilon) \qquad a \le r < a+\varepsilon \end{aligned} \tag{A12}$$

it follows that

$$I_1 = [(a+\varepsilon)^2] \,\Delta M_r(a+\varepsilon) - \tfrac{3}{2} \int_a^{a+\varepsilon} \Delta M_r r \,\mathrm{d}r + O(\varepsilon^3) \tag{A13}$$

whence, by condition (v) and (A12)

$$I_1 + I_2 = -\tfrac{3}{2} \int_a^{a+\varepsilon} \Delta M_r r \,\mathrm{d}r + O(\varepsilon^3). \tag{A14}$$

Since $\Delta M_\theta \le 0$ for $a \le r < a+\varepsilon$, $\Delta M_r(a+\varepsilon) < 0$ and (A10) is therefore established.

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Абстракт—Дается метод решения нелинейной задачи расчета на минимум объема сандвичевых осесимметрических пластинок, удовлетворяющих условию Мизеса. Этот способ является вариантом статического метода, при исполвзовании методов вариационного исчисления. Эти методы применяются для случая неограниченных минимальных решений. Даются разработка некоторых случаев для нагрузок, в которых приложнные усилия имеют тоже самое направление и опора находится только на внутренних или внешним краях. Указано, что характер расчета на миуимум объема не зависит от других особенностей таких нагрузок, когда пластинка оперта только на одном краю. Результаты работы сравниваются с подобными случаями пласминок, удовлетворяющих условию Трески.